Galois Descent

Riley Moriss

June 14, 2025

1 Statement

References are online notes by Kieth Conrad (2.14) or some guys online notes (2.2), Milne [Mil], [GW10, 14.20]. We claim that for finite K/k a Galois extension of fields we have an equivalence of categories

$$\mathrm{Vect}_k \leftrightarrow \mathrm{Vect}_K + \mathrm{Gal}(K/k) - \mathrm{actions}$$

Denote $\Gamma := Gal(K/k)$. To be precise, objects on the right are vector spaces over the algebraic closure V along with an action of the absolute Galois group on this vector space $\rho : \Gamma \to \mathrm{GL}(k, V)$ such that for $v, w \in V, c \in K$ it is Semi-linear:

$$\rho(g)(cv) = g(c)\rho(g)v$$

To be even more precise it is a family of maps $r_\sigma : V \to V$ for each $\sigma \in \Gamma$ satisfying

- r_σ is additive
- $r_\sigma(cv) = \sigma(c)r_\sigma(v)$
- $r_{id} = id$
- $r_\sigma \circ r_{\sigma'} = r_{\sigma\sigma'}$

Morphisms are algebra morphisms that respect the action. The equivalence is given by

$$- \otimes K : \mathrm{Vect}_k \leftrightarrow \mathrm{Vect}_K + \mathrm{Gal}(K/k) - \mathrm{actions} : (-)^\Gamma$$

On the right is taking the fixed points of the action. The action on the tensor is given by applying the Galois group element to the extension i.e.

$$V_k \mapsto V_K, \rho$$

where $\rho(\sigma)(v \otimes x) = v \otimes \sigma(x)$.

Remark: Note that the action is *not* K linear, as this would not encode any Galois information. It is semi-linear. In particular it is *not* a representation in the traditional sense. It does give a linear representation on $\mathrm{GL}(k, V)$, that is considering the K vector space as a k vector space, as the semi-linearity implies k linearity.

Remark: The infinite extension is true too but one must involve topologies to make the representations continuous.

Remark. This equivalence then propagates from vector spaces to algebras and then to all our algebraic categories, to affine schemes, and to the subcategory of LAG's and then reductive groups, their root datum and their Lie algebras. The point is that we can now send an algebra over $\bar{k}$ to an algebra over k by just picking a Galois action, and we can make precise why a classification over the algebraic closure is sufficient.

2 Real and Complex

Consider $\mathbb{C}/\mathbb{R}$ with $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$ where σ is conjugation. We first need to classify the Galois representations. By Maschke's theorem we have that real representations are also completely reducible and by the stack exchange gods the irreducible are given by the one (real) dimensional representations by $1 : \sigma \mapsto 1$ and $sgn : \sigma \mapsto -1$ Notice that the one dimensional reps are *not* semi-linear:

$$\sigma.(cv) = cv \neq \bar{c}v = \sigma(c)\sigma.v$$

$$\sigma.(cv) = -cv \neq -\bar{c}v = \sigma(c)\sigma.v$$

If we take their direct sum we get a one (complex) dimensional representation, $1 \oplus sgn$. Lets check that this is semi-linear. If $c = \begin{pmatrix} a \\ b \end{pmatrix}, v = \begin{pmatrix} u \\ w \end{pmatrix} \in \mathbb{C}$ then $cv = \begin{pmatrix} au - bw \\ aw + bu \end{pmatrix}$ and we have that

$$\sigma.(cu) = \begin{pmatrix} au - bw \\ -aw - bu \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix} \cdot \begin{pmatrix} u \\ -w \end{pmatrix} = \bar{c}\sigma.v$$

So we have two one real dimensional representations that we can put together to get a one complex dimensional representation. We need to look at complex "semi-linear" representations, and so they must be at least one complex dimensional to be in our category of complex vector spaces + Galois actions. Note *the direct sum is still a real representation* but it *is not a complex representation.* Notice that it is juts applying the conjugation to the complex number, this is not a complex representation but is *real.*

Now we have a complex vector space $\mathbb{C}$ with a semi-linear Galois action and we can look what its fixed points are. Because the action is just conjugation, flipping the sign of the second real coordinate, the fixed points are just the real numbers inside $\mathbb{C}$, that is the elements with zero in the second coordinate. So we get

$$\mathbb{C}^{1\oplus sgn} = \mathbb{R}$$

on the level of maps we have

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}, \mathbb{R}) = \mathbb{R}$$

what is $\mathrm{Hom}_{\mathbb{C}}\big((\mathbb{C}, 1\oplus sgn), (\mathbb{C}, 1\oplus sgn)\big)$, it is clear that $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ and because the intertwiners are in particular vector space homomorphisms we have the inclusion

$$\mathrm{Hom}_{\mathbb{C}}\big((\mathbb{C}, 1 \oplus sgn), (\mathbb{C}, 1 \oplus sgn)\big) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

So consider the map $\mathbb{C} \to \mathbb{C}$ given by multiplication by $c \in C$, we need to solve for which c we have $\forall v$

$$(1 \oplus sgn)(\sigma)(cv) = c(1 \oplus sgn)(\sigma)(v)$$

the LHS gives $\begin{pmatrix} au - bw \\ -aw - bu \end{pmatrix}$ as above and the right hand side gives $c\begin{pmatrix} u \\ -w \end{pmatrix} = \begin{pmatrix} au + bw \\ -aw + bu \end{pmatrix}$ solving these linear equations gives $\forall u, w \in \mathbb{R}$

$$-bw = bw, \quad -bu = bu$$

which is iff $b = 0$, i.e. this relation is true iff $c \in \mathbb{R}$. Hence we have shown that

$$\mathrm{Hom}_{\mathbb{C}}\big((\mathbb{C}, 1 \oplus sgn), (\mathbb{C}, 1 \oplus sgn)\big) = \mathbb{R}$$

and the functor is therefore clearly full and faithful!

References

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry I: Schemes With Examples and Exercises*. Vieweg+Teubner, Wiesbaden, 2010.

[Mil] J S Milne. Descent for algebraic schemes.

Also useful is
https://kconrad.math.uconn.edu/blurbs/galoistheory/galoisdescent.pdf
https://users.math.msu.edu/users/ruiterj2/math/Documents/Notes%20and%20talks/Galois%20descent.pdf
https://en.wikipedia.org/wiki/Maschke%27s_theorem
https://math.stackexchange.com/questions/219384/what-are-the-irreducible-representations-of-the-cycli